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The standard $(\mathfrak{g}, K)$-modules for $Sp(2, \mathbf{R})$, II

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概要

この小論は前回の論説 [?] に引き続いて $Sp(2,\mathbf{R})$ の標準 $(\mathfrak{g},K)$ 加群の記述を続ける．ここでは長いルートに付随する（極大）放物部分群 P_J によって誘導される一般化主系列表現を完全に明示的に記述する．

Abstract

As a continuation of the previous article [?], we continue the description of the standard $(\mathfrak{g},K)$-module structure of $Sp(2,\mathbf{R})$. Here we give a complete descrition of the generalized principal series associated with the parabolic subgroup P_J corresponding to the long root (we refer to such representations as the P_J principal series).

序

前回のノート [?] で、最も大きな主系列、つまり極小放物部分群から誘導される主系列表現の $(\mathfrak{g},K)$-加群を完全に明示的に記述した．注目に値するのは、結果を K の canonical basis (あるいは crystal basis) で記述するとき、shift operators ($\mathfrak{p}_\pm$ に関する gradient operators からくる微分作用素行列) や固有値の行列 (the matrices of generalized infinitesimal characters) が tri-diagonal や di-diagonal になることである．

Harish-Chandra の subquotient theorem や Casselman の subrepresentation theorem によれば、任意の既約 $(\mathfrak{g},K)$-加群は主系列 $(\mathfrak{g},K)$-加群の（特殊化の）組成剰余加群や部分加群として実現されるので、大まかに言えば、これで話は終わっている．しかし具体的なアルゴリズムとして、組成列や部分加群の構造を全て見るのは、やはり容易ではない．例えば、既約 $(\mathfrak{g},K)$-加群 π を主系列 Π_λ に埋入できたとして ($\pi \hookrightarrow \Pi_\lambda$), 各 K-type τ に対して π の中での τ-等質成分 (τ-isotypic component) $\pi[\tau]$ が Π_λ の τ-等質成分 $\Pi_\lambda[\tau]$ のどの一部に対応するのかは、双方の重複度が多きいときは一般には明らかでない．この種のことを少なくとも $G = Sp(2,\mathbf{R})$ のときに完全に記述するというのがこの企画の目標である．

ここでは長いルートに付随する放物部分群 P_J に関して誘導される表現 π を考える．この種の表現は Kostant-Vogan の意味で「大きい」(large), つまりその Gelfand-Kirillov dimension は $G = Sp(2,\mathbf{R})$ の極大ユニポテ

ント部分群 N の次元 $\dim N$ に等しい. しかしその associated variety の次数は 4 で、G のワイル群の位数の半分に等しい.

Introduction

In the previous note [?], we had an explicit description of the $(\mathfrak{g}, K)$-module structures of the largest principal series, i.e., the principal series induced via the minimal parabolic subgroup P_0. A notable fact is that in terms of the canonical basis (or the crystal basis) of the K-modules, the shift operators (the matrices of differential operators comming from the gradient operators with respect to $\mathfrak{p}_\pm$) and the matrices of eigenvalues (the generalized infinitesimal characters) are tri-diagonal, or di-diagonal.

By subquotient teorem of Harish-Chandra or by subrepresentation theorem of Casselman, any irreducible $(\mathfrak{g}, K)$-module is realized as a subquotient or submodules of a (specialization of a) principal series $(\mathfrak{g}, K)$-module. Therefore roughly speaking we have done the problem in the previous note [?]. However it is still difficult to grasp the structures of these subquotients and submodules, as concrete algorithm. For example, if we can embed an irreducible $(\mathfrak{g}, K)$-module π into a principal series Π_λ ($\pi \hookrightarrow \Pi_\lambda$), it is not clear in general that for each K-type τ how the τ-isotypic component $\pi[\tau]$ of π corresponds to a part of τ-isotypic component $\Pi_\lambda[\tau]$ of Π_λ, when the multiplicities of both are big.

The target of our project is to fix this kind of problem completely explicitly, at least, for the $(\mathfrak{g}, K)$-module strutures of $Sp(2, \mathbf{R})$.

Here we consider the representations π obtained by parabolic induction with respect to the parabolic subgroup P_J associated with the long root. These representations are *large* in the sense of Kostant-Vogan, i.e., their Gelfand-Kirillov dimension is equal to $\dim N$ of the maximal uipotent subgroup N of $G = Sp(2, \mathbf{R})$. But the degree of the associated variety is 4, i.e., the half of the order of the Weyl group of G.

1 The P_J-principal series

1.1 The parabolic subgroup P_J

The Lie algebra of the unipotent radical N_J of the maximal parabolic subgroup P_J associated with the subset $\{2e_2\}$ in the set $\{e_1 - e_2, 2e_2\}$ of simple roots of $Sp(2, \mathbf{R})$ is given by

$$\mathfrak{n}_J = \mathfrak{g}_{2e_1} \oplus \mathfrak{g}_{e_1 - e_2} \oplus \mathfrak{g}_{e_1 + e_2},$$

which is a Heisenberg algebra of dimension 3. We have

$$\mathfrak{a}_J = \mathbf{R} H_1, \qquad \text{with } H_1 = E_{11} - E_{33},$$

and $A_J = \exp(\mathfrak{a}_J)$.

The Levi part of the maximal parabolic subgroup P_J is given as a product $A_J M_J$ with

$$M_J = \left\{ \begin{pmatrix} \eta & & & \\ & a & & b \\ & & \eta^{-1} & \\ & c & & d \end{pmatrix} \mid \eta \in \mu_2 = \{\pm 1\}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}), \right\},$$

which is isomorphic to a direct product $\mu_2 \times SL(2, \mathbf{R})$. The connected component M_J^0 of M_J is isomorphic to $G_1 = SL(2, \mathbf{R})$.

We put $K_J := M_J \cap K$, then its connected component is given by $K_J^0 = M_J^0 \cap K$ which is isomorphic to

$$K_1 = \{ r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} | \theta \in \mathbf{R} \}$$

inside G_1.

1.2 Double coset decomposition for the P_J-principal series

For our later need, we want to have a decomposition of the standard basis $X_{\pm, ij}$ in $\mathfrak{p}_\pm$ with respect to

$$\mathrm{Ad}^{-1}(\mathfrak{n}_{J,\mathbf{C}} + \mathfrak{m}_{J,\mathbf{C}}) + \mathfrak{a}_{J,\mathbf{C}} + \mathfrak{k}_{\mathbf{C}}$$

for a given regular element $a_J \in A_J$. Since A_J centralizes $\mathfrak{m}_J$ and normalizes $\mathfrak{n}_J$, it suffices to know the following.

Lemma J.1 We have
(i) $X_{\pm,11} = \pm 2\sqrt{-1}E_{2e_1} + H_1 \pm \kappa(e_{11})$
with

$$\pm 2\sqrt{-1}E_{2e_1} \in \mathfrak{n}_J, \quad H_1 \in \mathfrak{a}_J, \quad \kappa(e_{11}) \in \mathfrak{k}_{\mathbf{C}};$$

(ii) $X_{+,12} = E_{e_1-e_2} + \sqrt{-1}E_{e_1+e_2} + \kappa(e_{21})$;
$X_{-,12} = E_{e_1-e_2} - \sqrt{-1}E_{e_1+e_2} - \kappa(e_{12})$
with

$$E_{e_1-e_2} \pm \sqrt{-1}E_{e_1+e_2} \in \mathfrak{n}_{J,\mathbf{C}} \text{ and } \kappa(e_{21}),\ \kappa(e_{12}) \in \mathfrak{k}_{\mathbf{C}};$$

(iii) $X_{\pm,22} = (\pm 2\sqrt{-1}E_{2e_2} + H_2) \pm \kappa(e_{22})$
with

$$\pm 2\sqrt{-1}E_{2e_2} + H_2 \in \mathfrak{m}_J \text{ and } \kappa(e_{22}) \in \mathfrak{k}_{\mathbf{C}}.$$

Moreover we note here that

$$\begin{aligned} \pm 2\sqrt{-1}E_{2e_2} + H_2 &= H_2 + \pm\sqrt{-1}(E_{2e_2} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) \pm \sqrt{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= x_\pm \pm \sqrt{-1}w, \end{aligned}$$

with

$$x_\pm = H_2 \pm \sqrt{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1.3 Definition of the P_J-principal series

The representation $\sigma_J \in \hat{M}_J$ is determined by a couple $\sigma_1 \in \widehat{SL(2,\mathbf{R})}$ and $\varepsilon \in \hat{\mu}_2$ as an outer tensor product $\sigma_J = \sigma_1 \otimes \varepsilon$. We assume that σ_1 is a representation of discrete series of $SL(2,\mathbf{R})$. Hence $\sigma_1 = D_k^+$ or $= D_k^-$ for some $k \geq 2$ ($k \in \mathbf{Z}$). Let $H_{D_k^\pm}$ be the representation of D_k^+ or of D_k^-, and denote by the same symbol its tensor product with the representation space $\mathbf{C}$ of ε over $\mathbf{C}$.

We fix a complex valued linear form ν_J on $\mathfrak{a}_J = \mathrm{Lie}(A_J)$ and let ρ_J be the half sum of roots belonging to $\mathfrak{n}_J$. The we consider a quasi-character

$$e^{\nu_J+\rho_J} : \ a_J \in A_J \mapsto \exp\{(\nu_J+\rho_J)(\log a_J)\} \in \mathbf{C}^\times .$$

Then the reprentation space H_π of the representation

$$\pi = \pi_{P_J;\nu_J,\sigma_J} = \mathrm{Ind}_{P_J}^G(\sigma_J \otimes e^{(\nu_J+\rho_J)} \otimes 1_{N_J})$$

of P_J principal series, is given by

$$\begin{aligned} H_\pi := \{ \ & \mathbf{f} : G \to H_{D_k^\pm}, \ \text{ locally integrable} \\ & \mathbf{f}(n_J m_J a_J x) = \sigma_J(m_J) e^{\nu_J+\rho_J)(\log a_J)} \mathbf{f}(x) \\ & \text{for a.e. } x \in G, n_J \in N_J, m_J \in M_J, a_J \in A_J \\ & \textstyle\int_K \|\mathbf{f}|K(x)\|^2_{H_{D_k^\pm}} dx < +\infty \} \end{aligned}$$

The action of G on this space is the right quasi-regular action given by

$$\pi(g) : \mathbf{f}(x) \mapsto \mathbf{f}(xg) \quad (x, g \in G, \mathbf{f} \in H_\pi).$$

1.4 The K-types of P_J principal series

We analyses the K-types of the representation space H_π of a P_J principal series. The target space of functions $\mathbf{f}$ in H_π has a decomposition:

$$H_{D_k^\pm} = \hat{\oplus}_{a=0}^{\infty} \mathbf{C} v_{\pm(k+2a)}.$$

Denote the corresponding decomposition of $\mathbf{f}$ by

$$\mathbf{f}(x) = \sum_{a=0}^{\infty} f_{\pm(k+2a)}(x) v_{\pm(k+2a)}.$$

Let $x \in K$, and let $m_J \in K_J$ be given by

$$m_J = (\eta, r_\theta) \text{ with } r_\theta \in K_1 \text{ and } \eta \in \mu_2$$

with respect to the isomorphism $K_J \cong \mu_2 \times SO(2)$. Then the defining relation of $\mathbf{f}$ reads

$$\mathbf{f}(k_J x) = \sigma_J(m_J)\mathbf{f}(x) = \varepsilon(\eta) D_k^\pm(r_\theta)\mathbf{f}(x) \quad (\text{a.e.} x \in K, k_J \in K_J).$$

The coefficients of v_m in the left hand side and the right hand side of this equality are given by

$$f_m((\eta, r_\theta)x) \text{ and } f_m(x)\varepsilon(\eta)e^{im\theta}, \text{ respectively.}$$

Therefore f_m belongs to

$$\begin{array}{rl} L^2_{(K_J,\varepsilon\otimes\chi_m)}(K) := \{ & f : G \to \mathbf{C}, \quad \text{locally integrable} \\ & f(k_J x) = \varepsilon(\eta)\chi_m(r_\theta)f(x) \\ & \text{for a.e. } x \in K, k_J = (\eta, r_\theta) \in K_J, \\ & \int_K |f|K(x)|^2 dx < +\infty\} \end{array}$$

for each $m = \pm(k+2a)$.

We recall here the equality of inner products:

$$\int_K \|\mathbf{f}|K(x)\|^2_{H_{D_k^\pm}} dx = \sum_m \{\int_K |f|K(x)|^2 dx\} \|v_m\|^2_{H_{D_k^\pm}}.$$

Proposition J.2 By restriction to K, we have an idenitification of spaces:

$$H_\pi = \widehat{\oplus}_{a=0}^\infty \{L^2_{(K_J,\varepsilon\otimes\chi_{\pm(k+2a)})}(K) \otimes v_{\pm(k+2a)}\}.$$

1.5 Construction of elementary functions

We construct elementary functions in

$$H_\pi = \widehat{\oplus}_{a=0}^\infty \{L^2_{(K_J,\varepsilon\otimes\chi_{\pm(k+2a)})}(K) \otimes_{\mathbf{C}} v_{\pm(k+2a)}\}.$$

First task is to contruct elementary functions in each factor $L^2_{(K_J,\varepsilon\otimes\chi_{\pm m})}(K)$ $(m = k+2a)$.

Lemma J.3 *The irreducible decompostion of $K \times K$ bimodule $L^2(K)$ induces an isomorphism of K-modules:*

$$\begin{aligned} L^2_{(K_J,\varepsilon\otimes\chi_{\pm m})}(K) &= \widehat{\oplus}_{\tau\in\hat{K}} \{(\tau^*|K_J)[\varepsilon \otimes \chi_{\pm m}] \boxtimes \tau\} \\ &= \widehat{\oplus}_{(l_1,l_2)\in L} \{(\tau^*_{(l_1,l_2)}|K_J)[\varepsilon \otimes \chi_{\pm m}] \boxtimes \tau_{(l_1,l_2)}\} \end{aligned}$$

Here τ^* is the contragradient representation of τ and $(\tau^*|K_J)[\varepsilon \otimes \chi_m]$ is the $\varepsilon \otimes \chi_m$-isotypic component in the K_J module $\tau^*|K_J$.

Proof. By definition, $L^2_{(K_J,\varepsilon\otimes\chi_{\pm m})}(K) = \{\mathbf{C}(\varepsilon \otimes \chi_m) \boxtimes L^2(K)\}^{K_J}$. Apply the decomposition of the $K \times K$-bimodules:

$$L^2(K) = \widehat{\bigoplus}_{(l_1,l_2)\in L}\tau^*_{(l_1,l_2)} \boxtimes \tau_{(l_1,l_2)}.$$

Since the operation to take the invaraint part with respect to K_J is involving only the first factor $\tau_{(l_1,l_2)}$ of each $\tau_{(l_1,l_2)}$-isotypic component, we have our proposition. □

Now we have to construct elementary functions in

$$(\tau^*_{(l_1,l_2)}|K_J)[\varepsilon \otimes \chi_{\pm m}] \boxtimes \tau_{(l_1,l_2)}.$$

The larger space $\tau^*_{(l_1,l_2)} \boxtimes \tau_{(l_1,l_2)}$ in $L^2(K)$ is generated by the entries in the matrix $\mathrm{Sym}^d(S(x))\det(S(x))^{l_2}$ ($x \in K, d = l_1 - l_2$). To see their intertwing properties with respect to the restriction to K_J of the left regular representation of K, we compute $\mathrm{Sym}^d(S(y))\det(S(y))^{l_2}$ for $y \in K_J$ given by

$$y = (\eta, r_\theta) \in \mu_2 \times SO(2),$$

which corresponds to $\begin{pmatrix} \eta & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ in $U(2)$.

Fortunately $\mathrm{Sym}^d(S(y))\det(S(y))^{l_2}$ ($y \in K_J$) is diagonalized as

$$\mathrm{diag}(\eta^d, \eta^{d-1}e^{i\theta}, \cdots, \eta^0 e^{i\theta})\eta^{l_2}e^{il_2\theta} = \mathrm{diag}(\eta^{l_1}e^{il_2\theta}, \eta^{l_1-1}e^{i(l_2+1)\theta}, \cdots, \eta^{l_2}e^{il_1\theta}).$$

Therefore the eigenspace of $\varepsilon \otimes \chi_{\pm m}(y) = \varepsilon(\eta)e^{\pm im\theta}$ occurs with multiplicity one, if there exists $a \in \mathbf{Z}$ ($0 \le a \le d$) such that

$$\eta^{l_1-a}e^{i(l_2+a)\theta} = \varepsilon(\eta)e^{\pm im\theta} \text{ for any } \eta \in \mu_2, \theta \in \mathbf{R}.$$

In this case, the eigenspace is generated by the $(d+1)$ entries the $(a+1)$-th row vector of the matrix $\mathrm{Sym}^d(S(x))\det(S(x))^{l_2}$ ($x \in K$). Thus we have the following.

Proposition J.4 The $\tau_{(l_1,l_2)}$-isotypic component of $L^2_{(K_J,\varepsilon\otimes\chi_{\pm m})}(K)$ is given by

$$L^2_{(K_J,\varepsilon\otimes\chi_{\pm m})}(K)(\tau_{(l_1,l_2)}) = \mathbf{C} < \mathbf{s}^{(d)}_{m-l_2}\Delta^{l_2} >,$$

if

$$l_2 \leq \pm m \leq l_1 \text{ and } \varepsilon(\eta) = \eta^{d\pm m} \text{ for any } \eta \in \mu_2.$$

Otherwise, it is $\{0\}$.

Corollary J.5 The natural surjection:

$$\widehat{\oplus}_{a=0}^{\infty} L^2_{(K_J,\varepsilon\otimes\chi_{\pm(k+2a)})}(K)v_{\pm(k+2a)} \twoheadrightarrow \left\{\sum_{a=0}^{\infty} L^2_{(K_J,\varepsilon\otimes\chi_{\pm(k+2a)})}(K)\right\}^{\frown} \subset L^2(K)$$

is an isomorphism.

Proof. This is obvious from the above construction of elementary functions.

Definition Let $\varepsilon \in \mu_2$ and $k \in \mathbf{Z}_{\geq 2}$. Then we put

$$M_pm(\varepsilon, k; (l_1, l_2)) := \{a \in \mathbf{Z}_{\geq 0} \mid l_2 \leq \pm(k+2a) \leq l_1, \text{ parity}(\varepsilon) \equiv l_1 - l_2 + k \pmod 2\}.$$

Proposition J.6 The $\tau_{(l_1,l_2)}$-isotypic component in $H_\pi = H_{\pi_{P_J,\nu_J,\sigma_J}}$ has a system of canonical basis consisting of vectors of elementary functions:

$$\{\mathbf{s}^{(l_1-l_2)}_{\pm(k+2a)-l_2}\Delta^{l_2} \mid a \in M_{\pm}(\varepsilon, k; (l_1, l_2)\}.$$

In particular, the multiplicity of $\tau_{(l_1,l_2)}$ in H_π is given by the cardinality

$$m(\pm; \varepsilon, k; (l_1, l_2)) = \# M_{\pm}(\varepsilon, k; (l_1, l_2))$$

of the set $M_{\pm}(\varepsilon, k; (l_1, l_2))$.

From now on, we consider only the $+$ sign in $\pm$. Therefore, the symbols $m(\pm; \varepsilon, k; (l_1, l_2))$ and $M_{\pm}(\varepsilon, k; (l_1, l_2))$ are abbridged to $m(\varepsilon, k; (l_1, l_2))$ and $M(\varepsilon, k; (l_1, l_2))$, respectively.

In order to formulate the corollary of the above proposition, we have to some notataion.

Notation (parity function and postive part)

(i) For $l \in \mathbf{Z}$, we define $p_k(l)$ by

$$p_k(l) := \begin{cases} 0, & \text{if } l \equiv k \textit{ mod } 2; \\ 1, & \text{if } l \not\equiv k \textit{ mod } 2. \end{cases}$$

(ii) For any function $f(l_1, l_2, k)$ in (l_1, l_2, k), we associate another function $f(l_1, l_2, k)_+$ by

$$f(l_1, l_2, k)_+ := sup\{0, f(l_1, l_2, k)\}.$$

Corollary J.7 The multiplicity $m(l_1, l_2)$ of $\tau_{(l_1,l_2)}$ in the principal series representation $\pi = \pi_{P_J,\nu_J,\sigma_J}$ is given as

$$\begin{aligned} m(l_1, l_2) &= \tfrac{1}{2}\{l_1 + 2 - p_k(l_1) - k\}_+ - \tfrac{1}{2}\{l_2 + p_k(l_2) - k\}_+ \\ &= \tfrac{1}{2}(sup\{l_1 - p_k(l_1),\ k\} - sup\{l_2 + p_k(l_2),\ k\}) + 1. \end{aligned}$$

2 Contiguous Relations for the P_J-principal series

2.1 Contiguous relations along the peripheral K-types

We investiagte the $(\mathfrak{g}, K)$-module structure on the subspace

$$L^2_{(K_J,\varepsilon\otimes\chi_k)} v_k \subset H_{(P_J,\nu_J,D_k^+\otimes\varepsilon)}$$

Inside this subspace there ocuurs K-types :

$$\begin{cases} \tau_{(k,k-2a)}(a = 0, 1, \cdots) & \text{for even case ;} \\ \tau_{(k,k-2a-1)}(a = 0, 1, \cdots) & \text{for odd cases} \end{cases}$$

with multiplicity one. Moreover eaxh $\tau_{(k,k-*)}$-isotopic component is given

$$\begin{cases} \text{either by} & < \mathbf{s}_{2a}^{(2a)} \Delta^{k-2a} > v_k \\ \text{or by} & < \mathbf{s}_{2a+1}^{2a+1} \Delta^{k-2a-1} > v_k \end{cases}$$

respectively.

Proposition J.8 (contiguous equations) We drop the generator v_k here.

(i) When $(-1)^k = \varepsilon$ (i.e., the even case), we have the *going-down* contiguous eqaution

$$\mathcal{C}^{(2a)}_{-,(+2)}\{\mathbf{s}_{2a}^{(2a)} \Delta^{k-2a}\} = \{\mathbf{s}_{2a+2}^{(2a+2)} \Delta^{k-2a-2}\}^* \gamma^{(2a)}_{-,(+2)}$$

with ${}^*\gamma^{(2a)}_{-,(+2)} = \nu_J + \rho_J - k + 2a$, and the *going-up* equation

$$\mathcal{C}^{(2a)}_{+,(-2)}\{\mathbf{s}^{(2a)}_{2a}\Delta^{k-2a}\} = \{\mathbf{s}^{(2a-2)}_{2a-2}\Delta^{k-2a+2}\}\gamma^{(2a)}_{+,(-2)}$$

with $\gamma^{(2a)}_{+,(-2)} = \nu_J + \rho_J + k - 2a - 2$.

(ii) When $(-1)^k = -\varepsilon$ (the odd case), we have

$$\mathcal{C}^{(2a+1)}_{-,(+2)}\{\mathbf{s}^{(2a+1)}_{2a+1}\Delta^{k-2a-1}\} = \{\mathbf{s}^{(2a+3)}_{2a+3}\Delta^{k-2a-3}\}{}^*\gamma^{(2a+1)}_{-,(+2)}$$

with the intertwining constant ${}^*\gamma^{(2a+1)}_{-,(+2)} = \nu_J + \rho - k + 2a + 1$, and

$$\mathcal{C}^{(2a+1)}_{+,(-2)}\{\mathbf{s}^{(2a+1)}_{2a+1}\Delta^{k-2a-1}\} = \{\mathbf{s}^{(2a-1)}_{2a-1}\Delta^{k-2a+1}\}\gamma^{(2a+1)}_{+,(-2)} \text{ (odd case)}$$

with the constant $\gamma^{(2a+1)}_{+,(-2)} = \nu_J + \rho_J + k - 2a - 3$.

Proof. (Going-down equations) Set $d = 2a$ or $d = 2a+1$, respectively. Since the value of $\mathbf{s}^{(d+2)}_{d+2}\Delta^{k-d-2}$ at e is the colum vector ${}^t(0, \cdots, 0, 1)$, it suffices to compute the value at e of $\mathcal{C}^{(d)}_{-;(+2)}\mathbf{s}^{(d)}_d\Delta^{k-d}$. The last row of $\mathcal{C}^{(d)}_{-;(+2)}$ is given by

$$(0, \cdots, 0, X_{-11}).$$

Therrefore, utilizing the double coset decomposition

$$X_{-11} = -2\sqrt{-1}E_{2e_1} + H_1 - \tau(e_{11}),$$

we have to compute the value

$$\begin{aligned} X_{-11}(s^d_{22}\Delta^{k-d})(e)v_k &= \{H_1(s_2 2^d\Delta^{k-d})(e) - \tau(e_{11})(s^d_{22}\Delta^{k-d})(e)\} \\ &= \{\nu_J + \rho_J - (k-d)\}v_k. \end{aligned}$$

(Going-up equations) Put $d = 2a$ or $d = 2a+1$. Then we have to compute the constant $\gamma^d_{+;(-2)}$ in

$$\mathcal{C}^{(d)}_{+;(-2)}\mathbf{s}^{(d)}_d\Delta^{k-d} = \gamma^d_{+;(-2)}\mathbf{s}^{d-2}_{d-2}\Delta^{k-d+2}.$$

Since the last row of the operator $\mathcal{C}^{(d)}_{+;(-2)}$ is

$$(0, \cdots, 0, X_{+22}, -2X_{+12}, X_{+11}),$$

we have to compute the sum of the three values

$$\begin{aligned}
X_{+22}(s_{21}^2 s_{22}^{d-2}\Delta^{k-d})(e) &= 0, \\
-2X_{+12}(s_{21}s_{22}^{d-1}\Delta^{k-d})(e) &= -2\kappa(e_{21})(s_{21}s_{22}^{d-1}\Delta^{k-d})(e) \\
&= -2(s_{22}^d\Delta^{k-d})(e) \\
&= -2,
\end{aligned}$$

and

$$X_{+11}(s_{22}^d\Delta^{k-d})(e) = (\nu_J + \rho_J) + (k-d).$$

Therefore $\gamma^d_{+;(-2)} = \nu_J + \rho_J + (k-d) - 2$. □

Remark (Comparison with the principal sereis) Let us specialize one of the parameters as $\nu_2 \mapsto k-1$, i.e. $\nu_2 + \rho_2 = k$ in the even P_{min}-principal series $\pi_{(P_{min};\nu_1,\nu_2,\varepsilon_1,\varepsilon_2)}$. Then its contiguous relation at the K-type $\tau_{(k,k)}$ ($\mathfrak{p}_-$-side, going-down) gives

$$\begin{pmatrix} X_{-22}(\Delta^k) \\ X_{-12}(\Delta^k) \\ X_{-11}(\Delta^k) \end{pmatrix} = (\nu_1 + 2 - k)\begin{pmatrix} s_{21}^2\Delta^{k-2} \\ s_{21}s_{22}\Delta^{k-2} \\ s_{22}^2\Delta^{k-2} \end{pmatrix}$$

This gives formally the same contiguous equation at $\tau_{(k,k)}$ of the even P_J-principal series, when we put $\nu_1 = \nu_J$. Needless to say, this corresponds to the exitence of the embedding of the P_J-principal series to the P_{min}-principal series. We already use this fact substancially in former papers.

2.2 Contiguous relations for general K-types

Before to discuss the contiguous relations, we compare firslty the multiplicities between contiguous K-types, i.e., the multiplicities

$$m(\varepsilon, k; (l_1, l_2)), \quad m(\varepsilon, k; (l_1', l_2'))$$

when $|l_1' - l_1| + |l_2' - l_2| = 2$, but $l_1 + l_2 \neq l_1' + l_2'$.

Claim J.9 Set

$$m = m(\varepsilon, k; (l_1, l_2)), \quad m' = m(\varepsilon, k; (l_1', l_2')).$$

Then in the case of even principal series representations, we have the following:

For $(l_1', l_2') = (l_1 + 2, l_2)$, $\quad m' = m + 1.$

For $(l_1', l_2') = (l_1, l_2 + 2)$, $\quad m' = \begin{cases} m & \text{if } l_2 < k \\ m - 1, & \text{if } l_2 \geq k. \end{cases}$

For $(l_1', l_2') = (l_1 + 1, l_2 + 1)$, $\quad m' = \begin{cases} m - 1 & \text{if } l_1 \equiv l_2 \equiv k \bmod 2 \text{ and } l_2 \geq k \\ m & \text{if } l_1 \equiv l_2 \equiv k \bmod 2 \text{ and } l_2 \leq k - 2 \\ m + 1, & \text{if } l_1 \equiv l_2 \equiv k + 1 \bmod 2. \end{cases}$

For $(l_1', l_2') = (l_1 - 2, l_2)$, $\quad m' = m - 1$

For $(l_1', l_2') = (l_1, l_2 - 2)$, $\quad m' = \begin{cases} m & \text{if } l_2 \leq k, \\ m + 1, & \text{if } l_2 > k. \end{cases}$

For $(l_1', l_2') = (l_1 - 1, l_2 - 1)$, $\quad m' = \begin{cases} m - 1 & \text{if } l_1 \equiv l_2 \equiv k \bmod 2 \\ m & \text{if } l_1 \equiv l_2 \not\equiv k \bmod 2 \text{ and } l_2 < k \\ m + 1, & \text{if } l_1 \equiv l_2 \not\equiv k \bmod 2 \text{ and } l_2 > k. \end{cases}$

In the case of odd principal series representations, we have

For $(l_1', l_2') = (l_1 + 2, l_2)$, $\quad m' = m + 1.$

For $(l_1', l_2') = (l_1, l_2 + 2)$, $\quad m' = \begin{cases} m & \text{if } l_2 \leq k - 2 \\ m - 1, & \text{if } l_2 \geq k - 1. \end{cases}$

For $(l_1', l_2') = (l_1 + 1, l_2 + 1)$, $\quad m' = \begin{cases} m & \text{if } l_1 \equiv k \bmod 2 \text{ or } l_2 \geq k; \\ m + 1, & \text{if } l_1 \not\equiv k \bmod 2 \text{ and } l_2 \leq k - 2. \end{cases}$

For $(l_1', l_2') = (l_1 - 2, l_2)$, $\quad m' = m - 1$

For $(l_1', l_2') = (l_1, l_2 - 2)$, $\quad m' = \begin{cases} m & \text{if } l_2 \leq k, \\ m + 1, & \text{if } l_2 \geq k + 1. \end{cases}$

For $(l_1', l_2') = (l_1 - 1, l_2 - 1)$, $\quad m' = \begin{cases} m & \text{if } l_1 \not\equiv k \bmod 2 \text{ or } l_2 \geq k \\ m - 1, & \text{if } l_1 \equiv k \bmod 2 \text{ and } l_2 < k. \end{cases}$

Now we have the canonical blocks of elementary functions, which are the source and target of the Dirac-Schmid operator considered in the section 1. Then we have the contiguous relations, if we determine the matrices of intertwining constants. To describe these results economically, we introduce more new symbols.

More local symbols For a dominat weight $(l_1,l_2)\in L_T^+$, using the former 'local' symbols:

$$p_k(l_i):=\begin{cases}0 & \text{if } l_i\equiv k \pmod 2\\ 1 & \text{if } l_i\not\equiv k \pmod 2\end{cases}\quad (i=1,2)$$

we set

$$\tilde{l}_1=l_1-p_k(l_1) \text{ and } \tilde{l}_2=l_2+p_k(l_2).$$

Proposition J.10.A
The case of even P_J-principal series $((-1)^k=\varepsilon)$.
Going-up relations:

(i) If $\tilde{l}_2\le k-2$, we have

$$\mathcal{C}_{+;(-2)}\mathbf{s}^{(d)}_{[k-l_2,\cdots,d-p_k(l_2)]}\Delta^{l_2}=\mathbf{s}^{(d-2)}_{[k-l_2-2,\cdots,d-2-p_k(l_2)]}\Delta^{l_2+2}\cdot\Gamma_{+;(-2)}(\pi_{P_J,k};(l_1,l_2))$$

with some $\frac{1}{2}\{\tilde{l}_1+2-k\}_+\times\frac{1}{2}\{\tilde{l}_1+2-k\}_+$ intertwining matrix $\Gamma_{+;(-2)}(\pi_{P_J,k};(l_1,l_2))$ which is given by

$$\begin{aligned}\Gamma_{+;(-2)}(\pi_{P_J,k};(l_1,l_2)) &= \begin{bmatrix}\mathbf{0} & , & 0\\ \mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2)\le i\le\frac{d}{2}-1-p_k(l_1)}(k+\tilde{l}_2+2i) & , & \mathbf{0}\end{bmatrix}\\ &\quad+\mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2-2)\le i\le\frac{d}{2}-1-p_k(l_1)}(\nu_J+\rho_J+\tilde{l}_2-d+2i).\end{aligned}$$

(ii) If $\tilde{l}_2\ge k$, we have

$$\mathcal{C}_{+;(-2)}\mathbf{s}^{(d)}_{[0,\cdots,d-p_k(l_2)]}\Delta^{l_2}=\mathbf{s}^{(d-2)}_{[0,\cdots,d-2-p_k(l_2)]}\Delta^{l_2+2}\cdot\Gamma_{+;(-2)}(\pi_{P_J,k};(l_1,l_2))$$

with some $(\frac{d}{2}-p_k(l_1))\times(\frac{d}{2}+1-p_k(l_1))$ matrix $\Gamma_{+;(-2)}(\pi_{P_J,k};(l_1,l_2))$, given by

$$\begin{aligned}\Gamma_{+;(-2)}(\pi_{P_J};(l_1,l_2)) &= \left[\mathbf{diag}_{0\le i\le\frac{d}{2}-1-p_k(l_1)}(k+\tilde{l}_2+2i),\mathbf{0}\right]\\ &\quad+\left[\mathbf{0},\ \mathbf{diag}_{0\le i\le\frac{d}{2}-1-p_k(l_1)}(\nu_J+\rho_J+\tilde{l}_2-d+2i)\right].\end{aligned}$$

Slant-up relations: $\mathcal{C}_{+;(0)}:\tau_{(l_1,l_2)}\to\tau_{(l_1+1,l_2+1)}$ (3 cases)

(iii) If $p_k(l_1)=p_k(l_2)=0$ and $l_2\le k-2$, we have $m'=m$ and

$$\mathcal{C}_{+;(0)}\mathbf{s}^{(d)}_{[k-l_2,\cdots,d]}\Delta^{l_2}=\mathbf{s}^{(d)}_{[k-l_2-1,\cdots,d-1]}\Delta^{l_2+1}\cdot\Gamma_{+;(0)}(\pi_{P_J,k};(l_1,l_2))$$

with some $\frac{1}{2}\{l_1+2-k\}_+ \times \frac{1}{2}\{l_1+2-k\}_+$ intertwining matrix $\Gamma_{+;(0)}(\pi_{P_J,k};(l_1,l_2))$, which is given by

$$\begin{aligned}\Gamma_{+;(0)}(\pi_{P_J,k};(l_1,l_2)) &= \begin{bmatrix} \mathbf{0}, & 0 \\ \mathbf{diag}_{1\le i\le \frac{l_1-(k-2)}{2}-1}(-\frac{2i+1}{d}(k+l_2+2i), & \mathbf{0}\end{bmatrix} \\ &+ \mathbf{diag}_{0\le i\le \frac{l_1-(k-2)}{2}-1}(\frac{d-2i-1}{d}(\nu_J+\rho_J+l_2+2i)).\end{aligned}$$

(iv) If $p_k(l_1)=p_k(l_2)=0$ and $l_2\ge k$, we have $m'=m-1$ and

$$\mathcal{C}_{+;(0)}\mathbf{s}^{(d)}_{[0,\cdots,d]}\Delta^{l_1} = \mathbf{s}^{(d)}_{[1,\cdots,d-1]}\Delta^{l_2+1}\cdot\Gamma$$

with some $\frac{d}{2}\times(\frac{d}{2}+1)$ intertwining matrix Γ, which is given by

$$\begin{aligned}\Gamma_{+;(0)}(\pi_{P_J,k};(l_1,l_2)) &= \left[\mathbf{diag}_{0\le i\le \frac{d}{2}-1}\left(-\frac{2i+1}{d}(k+l_2+2i)\right), \mathbf{0}\right] \\ &+ \left[\mathbf{0},\ \mathbf{diag}_{0\le i\le \frac{d}{2}-1}\left(\frac{d-2i-1}{d}(\nu_J+\rho_J+l_2+2i)\right)\right].\end{aligned}$$

(v) If $p_k(l_1)=p_k(l_2)=1$, we have $m'=m+1$ and

$$\mathcal{C}_{+;(0)}\mathbf{s}^{(d)}_{[sup\{k-l_2,\ 1\},\cdots,d-1]}\Delta^{l_2} = \mathbf{s}^{(d)}_{[sup\{k-l_2-1,0\},\cdots,d]}\Delta^{l_2+1}\cdot\Gamma_{+;(0)}(\pi_{P_J,k};(l_1,l_2))$$

with some $\frac{1}{2}(\tilde{l}_1+4-sup\{k,\tilde{l}_2\})\times\frac{1}{2}(\tilde{l}_1+2-sup\{k,\tilde{l}_2\})$ intertwining matrix $\Gamma_{+;(0)}(\pi_{P_J,k};(l_1,l_2))$, which is given by

$$\begin{aligned}&\Gamma_{+;(0)}(\pi_{P_J,k};(l_1,l_2)) \\ &= \begin{bmatrix}\mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2)\le i\le\frac{1}{2}(d-2)}(\frac{d-2i-2}{d}(\nu_J+\rho_J+l_2+2i-1)) \\ \mathbf{0}\end{bmatrix} \\ &+ \begin{bmatrix}\mathbf{0} \\ \mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2)\le i\le\frac{1}{2}(d-2)}(-\frac{2i+2}{d}(k+l_2+2i+1))\end{bmatrix}.\end{aligned}$$

Going-right relation: $\mathcal{C}_{+;(+2)}:\tau_{(l_1,l_2)}\to\tau_{(l_1+2,l_2)}$

(vi) We have,

$$\begin{aligned}&\mathcal{C}_{+;(+2)}\mathbf{s}^{(d)}_{[sup\{k-l_2,\ p_k(l_2)\},\cdots,d-p_k(l_2)]}\Delta^{l_2} = \\ &\mathbf{s}^{(d+2)}_{[sup\{k-l_2,\ p_k(l_2)\},\cdots,d+2-p_k(l_2)]}\Delta^{l_2}\cdot\Gamma_{+;(+2)}(\pi_{P_J,k};(l_1,l_2))\end{aligned}$$

with some

$$\frac{1}{2}(\tilde{l}_1-sup\{\tilde{l}_2,k\})\times\frac{1}{2}(\tilde{l}_1+2-sup\{\tilde{l}_2,k\})$$

intertwining matrix $\Gamma_{+;(+2)}(\pi_{P_J,k};(l_1,l_2))$, which is given by

$$\Gamma_{+;(+2)}(\pi_{P_J,k};(l_1,l_2))$$

$$= \begin{bmatrix} \mathbf{0} \\ \mathbf{diag}_{sup\{\frac{1}{2}(k-\tilde{l}_2,0\}\leq i\leq \frac{d}{2}-p_k(l_2)} \left(\frac{(2i+1+p_k(l_2))(2i+2+p_k(l_2))}{(d+1)(d+2)}(k+\tilde{l}_2+2i)\right), \end{bmatrix}$$
$$+ \begin{bmatrix} \mathbf{diag}_{sup\{\frac{1}{2}(k-\tilde{l}_2,0\}\leq i\leq \frac{d}{2}-p_k(l_2)} \left(\frac{(d+1-2i-p_k(l_2))(d+2-2i-p_k(l_2))}{(d+1)(d+2)}(\nu_J+\rho_J+\tilde{l}_2+d+2i)\right) \\ \mathbf{0} \end{bmatrix}$$

Going-left relations: $\mathcal{C}_{-;(-2)} : \tau_{(l_1,l_2)} \to \tau_{(l_1-2,l_2)}$

(i) We have $m' = m-1$ and

$$\mathcal{C}_{-;(-2)}\mathbf{s}^{(d)}_{[sup\{k-l_2,p_k(l_2)\},\cdots,d-p_k(l_2)]}\Delta^{l_2} = \mathbf{s}^{(d-2)}_{[sup\{k-l_2,p_k(l_2)\},\cdots,d-2-p_k(l_2)]}\Delta^{l_2}\cdot\Gamma_{-;(-2)}(\pi_{P_J,k};(l_1,l_2))$$

with some $\frac{1}{2}(\tilde{l}_1 - sup\{k,\tilde{l}_2\}) \times \frac{1}{2}(\tilde{l}_1 + 2 - sup\{k,\tilde{l}_2\})$ intertwining matrix $\Gamma_{-;(-2)}(\pi_{P_J,k};(l_1,l_2))$, which is given by

$$\Gamma_{-;(-2)}(\pi_{P_J,k};(l_1,l_2))$$

$$= \left[\mathbf{diag}_{sup\{\frac{1}{2}(k-\tilde{l}_2),0\}\leq i\leq \frac{d}{2}-1-p_k(l_2)}\left((\nu_J+\rho_J)-(d+\tilde{l}_2+2i+2)\right),\mathbf{0}\right]$$
$$+\left[\mathbf{0},\ \mathbf{diag}_{sup\{\frac{1}{2}(k-\tilde{l}_2),0\}\leq i\leq \frac{d}{2}-1-p_k(l_2)}(k-\tilde{l}_2-2i-2)\right].$$

Note here that we have $\frac{d}{2}-1-p_k(l_2) = \frac{1}{2}(\tilde{l}_1-\tilde{l}_2)-1$.
Slant-down relations: $\mathcal{C}_{-;(0)} : \tau_{(l_1,l_2)} \to \tau_{(l_1-1,l_2-1)}$ (3 cases)

(ii) When $p_k(l_1) = p_k(l_2) = 0$, we have $m' = m-1$ and

$$\mathcal{C}_{-;(0)}\mathbf{s}^{(d)}_{[sup\{k-l_2,0\},\cdots,d]}\Delta^{l_1} = \mathbf{s}^{(d)}_{[sup\{k-l_2-1,1\},\cdots,d-1]}\Delta^{l_2-1}\cdot\Gamma_{-;(0)}(\pi_{P_J};(l_1,l_2))$$

with some $\frac{1}{2}(d+1-sup\{k-l_2+1,1\}) \times \frac{1}{2}(d+2-sup\{k-l_2,0\})$ intertwining matrix $\Gamma_{-;(0)}(\pi_{P_J};(l_1,l_2))$, which is given by

$$\begin{aligned}\Gamma_{-;(0)}(\pi_{P_J};(l_1,l_2)) &= \left[\mathbf{diag}_{\frac{1}{2}sup\{k-l_2,0\}\leq i\leq\frac{d}{2}-1}\left(\tfrac{2i+1}{d}(\nu_J+\rho_J)-(l_2+2i+2)\right),\mathbf{0}\right] \\ &+\left[\mathbf{0},\ \mathbf{diag}_{\frac{1}{2}sup\{k-l_2,0\}\leq i\leq\frac{d}{2}-1}\left(\tfrac{d-1-2i}{d}(k-l_2-2i-2)\right)\right].\end{aligned}$$

(iii) If $p_k(l_1) = p_k(l_2) = 1$ and $l_2 \leq k-3$, we have $m' = m$ and

$$\mathcal{C}_{-;(0)} \mathbf{s}^{(d)}_{[k-l_2,\cdots,d-1]} \Delta^{l_2} = \mathbf{s}^{(d)}_{[k-l_2+1,\cdots,d]} \Delta^{l_2-1} \cdot \Gamma_{-;(0)}(\pi_{P_J,k}; (l_1, l_2))$$

with some $\frac{l_1-k+1}{2} \times \frac{l_1-k+1}{2}$ intertwining matrix $\Gamma_{-;(0)}(\pi_{P_J,k}; (l_1, l_2))$, which is given by

$$\Gamma_{-;(0)}(\pi_{P_J,k}; (l_1, l_2)) = \begin{bmatrix} \mathbf{0}, & \mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2)+1 \leq i \leq \frac{d}{2}-1}(\frac{d-2i}{d}(k-2i-\tilde{l}_2)) \\ 0, & \mathbf{0} \end{bmatrix}$$
$$+\mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2)+1 \leq i \leq \frac{d}{2}-1}(-\tfrac{2i+2}{d}(\nu_j + \rho_J - \tilde{l}_2 - 2i - 2))$$

(iv) If $p_k(l_1) = p_k(l_2) = 1$ and $l_2 \geq k-1$, we have $m' = m+1$ and

$$\mathcal{C}_{-;(0)} \mathbf{s}^{(d)}_{[1,\cdots,d-1]} \Delta^{l_2} = \mathbf{s}^{(d)}_{[0,\cdots,d]} \Delta^{l_2-1} \cdot \Gamma_{-;(0)}(\pi_{P_J,k}; (l_1, l_2))$$

with some $(\frac{d}{2}+1) \times \frac{d}{2}$ intertwining matrix $\Gamma_{-;(0)}(\pi_{P_J,k}; (l_1, l_2))$, which is given by

$$\Gamma_{-;(0)}(\pi_{P_J,k}; (l_1, l_2)) = \begin{bmatrix} \mathbf{diag}_{0 \leq i \leq \frac{d}{2}-1}\left(\frac{d-2i}{d}(k-l_2-2i-1)\right) \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0}, \\ \mathbf{diag}_{0 \leq i \leq \frac{d}{2}-1}\left(-\frac{2i+2}{d}(\nu_J + \rho_J - l_2 - 2i - 3)\right) \end{bmatrix}.$$

Going-down relations: $\mathcal{C}_{-;(+2)} : \tau_{(l_1,l_2)} \to \tau_{(l_1,l_2-2)}$ (two cases)

(v) When $\tilde{l}_2 \leq k$, we have

$$\mathcal{C}_{-;(+2)} \mathbf{s}^{(d)}_{[k-l_2,\cdots,d-p_k(l_2)]} \Delta^{l_2} = \mathbf{s}^{(d+2)}_{[k-l_2+2,\cdots,d+2-p_k(l_2)]} \Delta^{l_2-2} \cdot \Gamma_{-;(+2)}(\pi_{P_J,k}; (l_1, l_2))$$

with some $(\frac{\tilde{l}_1-k+2}{2}) \times (\frac{\tilde{l}_1-k+2}{2})$ intertwining matrix $\Gamma_{-;(+2)}(\pi_{P_J,k}; (l_1, l_2))$, which is given by $(m' = m)$

$$\Gamma_{-;(+)}(\pi_{P_J,k}; (l_1, l_2))$$
$$= \begin{bmatrix} \mathbf{0}, & \mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2+2) \leq i \leq \frac{d}{2}-p_k(l_2)}\left(\frac{(d+1-p_k(l_2)-2i)(d+2-p_k(l_2)-2i)}{(d+1)(d+2)}(k-\tilde{l}_2-2i)\right) \\ 0, & \mathbf{0} \end{bmatrix}$$
$$+\mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2) \leq i \leq \frac{d}{2}-p_k(l_2)}\left(\frac{(2i+1+p_k(l_2))(2i+2+p_k(l_2))}{(d+1)(d+2)}(\nu_J + \rho_J - \tilde{l}_2 - 2i + d)\right).$$

(vi) When $\tilde{l}_2 \geq k+2$, we have

$$\mathcal{C}_{-;(+2)}\mathbf{s}^{(d)}_{[p_k(l_2),\cdots,d-p_k(l_2)]}\Delta^{l_2} = \mathbf{s}^{(d+2)}_{[p_k(l_2),\cdots,d+2-p_k(l_2)]}\Delta^{l_2-2}\cdot\Gamma_{-;(+)}(\pi_{P_J,k};(l_1,l_2))$$

with some $(\frac{d}{2}-p_k(l_2))\times(\frac{d}{2}+1-p_k(l_2))$ intertwining matrix $\Gamma_{-;(+)}(\pi_{P_J,k};(l_1,l_2))$, which is given by

$$\begin{aligned}&\Gamma_{-;(+)}(\pi_{P_J,k};(l_1,l_2))\\ =&\begin{bmatrix}\mathbf{diag}_{0\leq i\leq\frac{d}{2}-p_k(l_2)}\left(\frac{(d+1-p_k(l_2)-2i)(d+2-p_k(l_2)-2i)}{(d+1)(d+2)}(k-\tilde{l}_2-2i)\right)\\ \mathbf{0}\end{bmatrix}\\ +&\begin{bmatrix}\mathbf{0}\\ \mathbf{diag}_{0\leq i\leq\frac{d}{2}-p_k(l_2)}\left(\frac{(2i+1+p_k(l_2))(2i+2+p_k(l_2))}{(d+1)(d+2)}(\nu_J+\rho_J-\tilde{l}_2-2i+d)\right)\end{bmatrix}.\end{aligned}$$

Next we discuss the case of odd P_J's. We define the micro-parity of each K-type $\tau_{(l_1,l_2)}$.

Definition If $((-1)^{l_1},(-1)^{l_2})=(\varepsilon,(-1)^k)$ (resp. $((-1)^{l_1},(-1)^{l_2})=(-\varepsilon,-(-1)^k)$), we say that the *miro-parity* of (l_1,l_2) is $+1$ (resp. -1) (or even (resp. odd)).

Proposition J.10.B

The case of odd P_J-principal series ($(-1)^k=-\varepsilon$). In this case only those K-types $\tau_{(l_1,l_2)}$ such that $d=l_1-l_2$ is odd occur.

Going-up relations: $\mathcal{C}_{+;(-2)}:\tau_{(l_1,l_2)}\to\tau_{(l_1,l_2+2)}$ (two cases)

(i) If $\tilde{l}_2\leq k-2$, we have

$$\mathcal{C}_{+;(-2)}\mathbf{s}^{(d)}_{[k-l_2,\cdots,d-p_k(l_1)]}\Delta^{l_2} = \mathbf{s}^{(d-2)}_{[k-l_2-2,\cdots,d-2-p_k(l_1)]}\Delta^{l_2+2}\cdot\Gamma_{+;(-2)}(\pi_J,k;(l_1,l_2))$$

with some $\frac{l_1-k}{2}\times\frac{l_1-k}{2}$ intertwining matrix $\Gamma_{+;(-2)}(\pi_J,k;(l_1,l_2))$, which is given by

$$\begin{aligned}&\Gamma_{+;(-2)}(\pi_J,k;(l_1,l_2))\\ =&\begin{bmatrix}\mathbf{0}, & 0\\ \mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2+2)\leq a\leq(d-3)/2}(k+\tilde{l}_2+2a), & \mathbf{0}\end{bmatrix}\\ &+\mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2)\leq a\leq(d-3)/2}(\nu_J+\rho_J+\tilde{l}_2-d+2a).\end{aligned}$$

(ii) If $\tilde{l}_2\geq k$, we have

$$\mathcal{C}_{+;(-2)}\mathbf{s}^{(d)}_{[p_k(l_2),\cdots,d-p_k(l_1)]}\Delta^{l_2} = \mathbf{s}^{(d-2)}_{[p_k(l_2),\cdots,d-2-p_k(l_1)]}\Delta^{l_2+2}\cdot\Gamma_{+;(0)}(\pi_J,k;(l_1,l_2))$$

with some $(\frac{d-1}{2}) \times (\frac{d+1}{2})$ intertwining matrix $\Gamma_{+;(0)}(\pi_J, k; (l_1, l_2))$, which is given by

$$\begin{aligned}\Gamma_{+;(-2)}(l_1,l_2) &= \left[(\mathbf{diag}_{0\le a\le (d-3)/2}(k+\tilde{l}_2+2a), \mathbf{0})\right] \\ &+ \left[\mathbf{0}, \mathbf{diag}_{0\le a\le (d-3)/2}(\nu_J+\rho_J+\tilde{l}_2-d+2a)\right].\end{aligned}$$

Slant-up relations: $\mathcal{C}_{+;(0)} : \tau_{(l_1,l_2)} \to \tau_{(l_1+1,l_2+1)}$ (3 cases)

(iii) If micro-parity is odd, we have $m' = m$ and

$$\mathcal{C}_{+;(0)}\mathbf{s}^{(d)}_{[sup\{k-l_2,1\},\cdots,d]}\Delta^{l_2} = \mathbf{s}^{(d)}_{[sup\{k-l_2-1,0\},\cdots,d-1]}\Delta^{l_2+1}\cdot\Gamma_{\pi_{P_J},k}(l_1,l_2)$$

with some $\frac{1}{2}(l_1-k+2)\times\frac{1}{2}(l_1-k+2)$ intertwining matrix $\Gamma_{+;(0);\pi_{P_J},k}(l_1,l_2)$, which is given by

$$\begin{aligned}\Gamma_{+;(0);10}(l_1,l_2) &= \mathbf{diag}_{sup\{\frac{1}{2}(k-l_2-1),0\}\le a\le\frac{d-1}{2}}\left(\frac{d-2a}{d}(\nu_J+\rho_J+l_2+2a-1)\right) \\ &+ \begin{bmatrix} \mathbf{0}, & 0 \\ \mathbf{diag}_{sup\{\frac{1}{2}(k-l_2+1),1\}\le a\le\frac{d-1}{2}}(-\frac{2a}{d}(k+l_2+2a-1)), & \mathbf{0}\end{bmatrix}\end{aligned}$$

(iv) If the micro-parity of (l_1,l_2) is even, and $l_2 \ge k$, we have $m' = m$ and

$$\mathcal{C}_{+;(0)}\mathbf{s}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_1} = \mathbf{s}^{(d)}_{[1,\cdots,d]}\Delta^{l_2+1}\cdot\Gamma$$

with some $\frac{d+1}{2}\times\frac{d+1}{2}$ intertwining matrix Γ, which is given by

$$\begin{aligned}\Gamma_{+;(0);01}(l_1,l_2) &= \mathbf{diag}_{0\le a\le\frac{d-1}{2}}\left(-\frac{2a+1}{d}(k+\rho_2+l_2+2a)\right) \\ &+ \begin{bmatrix} \mathbf{0}, & \mathbf{diag}_{0\le a\le\frac{d-3}{2}}(\frac{d-2a-1}{d}(\nu_J+\rho_J+l_2+2a)) \\ 0, & \mathbf{0}\end{bmatrix}\end{aligned}$$

(v) If the micro-parity of (l_1,l_2) is even and $l_2 \le k-2$ we have $m' = m+1$

$$\mathcal{C}_{+;(0)}\mathbf{s}^{(d)}_{[k-l_2,\cdots,d-1]}\Delta^{l_2} = \mathbf{s}^{(d)}_{[k-l_2-1,\cdots,d]}\Delta^{l_2+1}\cdot\Gamma_{+;(0);\pi_{P_j},k}(l_1,l_2)$$

with some $\frac{l_1-k-1}{2}\times\frac{l_1-k+1}{2}$ intertwining matrix $\Gamma_{+;(0)\pi_{P_j},k}(l_1,l_2)$, which is given by

$$\begin{aligned}\Gamma_{+;(0);01}(l_1,l_2) &= \begin{bmatrix} \mathbf{0} \\ \mathbf{diag}_{\frac{1}{2}(k-l_2+2)\le a\le\frac{d-1}{2}}(-\frac{2a+1}{d}(k+l_2+2a))\end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{diag}_{\frac{1}{2}(k-l_2)\le a\le\frac{d-3}{2}}(\frac{d-2a-1}{d}(\nu_J+\rho_J+l_2+2a)) \\ \mathbf{0}\end{bmatrix}\end{aligned}$$

Going-right relations: $\mathcal{C}_{+;(+2)}:\tau_{(l_1,l_2)}\to\tau_{(l_1+2,l_2)}$ (one case)

(vi) We have $m'=m+1$ and

$$\mathcal{C}_{+;(+2)}\mathbf{s}^{(d)}_{[sup\{k-l_2,p_k(l_2)\},\cdots,d-p_k(l_1)]}\Delta^{l_2}=\mathbf{s}^{(d+2)}_{[sup\{k-l_2,p_k(l_2)\},\cdots,d+2-p_k(l_1)]}\Delta^{l_2}\cdot\Gamma_{+;(+2)}(l_1,l_2)$$

with some $(\frac{\tilde{l}_1+2-sup\{k,\tilde{l}_2\}}{2})\times(\frac{\tilde{l}_1+4-sup\{k,\tilde{l}_2\}}{2})$ intertwining matrix $\Gamma_{+;(+2)}(l_1,l_2)$, which is given by

$$\begin{aligned}\Gamma_{+;(+2)}(l_1,l_2) &= \begin{bmatrix}\mathbf{diag}_{\frac{1}{2}\{k-\tilde{l}_2\}_+\le a\le\frac{d-1}{2}}\left(\frac{(d-2a)(d+1-2a)}{d+1)(d+2)}(\nu_J+\rho_J+\tilde{l}_2+d+2a)\right)\\ \mathbf{0}\end{bmatrix}\\ &+\begin{bmatrix}\mathbf{0}\\ \mathbf{diag}_{\frac{1}{2}\{k-\tilde{l}_2\}_+\le a\le\frac{d-1}{2}}\left(\frac{(2a+2)(2a+3)}{(d+1)(d+2)}(k+\tilde{l}_2+2a)\right)\end{bmatrix}\end{aligned}$$

Going-left relations: $\mathcal{C}_{-;(-2)}:\tau_{(l_1,l_2)}\to\tau_{(l_1-2,l_2)}$: (one case only)

(i) We have $m'=m-1$ and

$$\mathcal{C}_{-;(-2)}\mathbf{s}^{(d)}_{[sup\{k-l_2,p_k(l_2)\},\cdots,d-p_k(l_1)]}\Delta^{l_1}=\mathbf{s}^{(d-2)}_{[sup\{k-l_2,p_k(l_2)\},\cdots,d-2-p_k(l_1)]}\Delta^{l_1}\cdot\Gamma_{-;(-2)}$$

with some $\frac{1}{2}(\tilde{l}_1-sup\{k,\tilde{l}_2\})\times\frac{1}{2}(\tilde{l}_1-sup\{k,\tilde{l}_2\}+2)$ intertwining matrix $\Gamma_{-;(-2)}$, which is given by

$$\begin{aligned}\Gamma_{-;(-2)}(l_1,l_2) &= \left[(\mathbf{diag}_{0\le a\le(d-3)/2}\{(\nu_J+\rho_J)-(\tilde{l}_2+d+2a+2)\},\quad\mathbf{0})\right]\\ &+\left[\mathbf{0},\quad\mathbf{diag}_{0\le a\le(d-3)/2}\{k-(\tilde{l}_2+2a+2)\}\right].\end{aligned}$$

Slant-down relations: $\mathcal{C}_{-;(0)}:\tau_{(l_1,l_2)}\to\tau_{(l_1-1,l_2-1)}$ (3 cases)

(ii) If (l_1,l_2) has odd micro-parity with $l_2\le k-1$, we have $m'=m-1$ and

$$\mathcal{C}_{-;(0)}\mathbf{s}^{(d)}_{[k-l_2,\cdots,d]}\Delta^{l_2}=\mathbf{s}^{(d)}_{[k-l_2+1,\cdots,d-1]}\Delta^{l_2-1}\cdot\Gamma_{-;(0)}$$

with some $\frac{l_1-k}{2}\times\frac{l_1-k+2}{2}$ intertwining matrix $\Gamma_{-;(0)}$, which is given by

$$\begin{aligned}\Gamma_{-;(0);10}(l_1,l_2) &= \left[\mathbf{diag}_{\frac{1}{2}(k-l_2+1)\le a\le\frac{d-1}{2}}\left(\frac{d-2a}{d}\{k-(l_2+2a+1)\}\right)\right]\\ &+\left[\mathbf{diag}_{1\le a\le\frac{d-1}{2}}\left(-\frac{2a+2}{d}\{(\nu_J+\rho_J)-(l_2+2a+3)\}\right)\quad\mathbf{0}\right]\end{aligned}$$

(iii) If (l_1, l_2) has odd micro-parity with $l_2+1 \geq k$, we have $m' = m$ and

$$\mathcal{C}_{-;(0)}\mathbf{s}^{(d)}_{[1,\cdots,d]}\Delta^{l_2} = \mathbf{s}^{(d)}_{[0,\cdots,d-1]}\Delta^{l_2-1} \cdot \Gamma_{-;(0)}$$

with some $\frac{d+1}{2} \times \frac{d+1}{2}$ intertwining matrix $\Gamma_{-;(0)}$, which is given by

$$\begin{aligned}\Gamma_{-;(0);10}(l_1,l_2) &= \mathbf{diag}_{0\leq a\leq\frac{d-1}{2}}\left(\frac{d-2a}{d}\{k-(l_2+2a+1)\}\right)\\ &+ \begin{bmatrix} \mathbf{0}, & 0 \\ \mathbf{diag}_{1\leq a\leq\frac{d-1}{2}}\left(-\frac{2a+2}{d}\{(\nu_J+\rho_J)-(l_2+2a+3)\}\right) & \mathbf{0}\end{bmatrix}\end{aligned}$$

(iv) If (l_1, l_2) has even micro-parity, we have $m' = m$ and

$$\mathcal{C}_{-;(0)}\mathbf{s}^{(d)}_{[sup\{k-l_2,0\},\cdots,d-1]}\Delta^{l_2} = \mathbf{s}^{(d)}_{[sup\{k-l_2+1,1\},\cdots,d]}\Delta^{l_2-1} \cdot \Gamma_{-;(0);***}$$

with some $(\frac{l_1-sup\{k,l_2\}+1}{2})\times(\frac{l_1-sup\{k,l_2\}+1}{2})$ intertwining matrix $\Gamma_{-;(0);***}$, which is given by

$$\begin{aligned}\Gamma_{-;(0);01}(l_1,l_2) &= \mathbf{diag}_{\frac{1}{2}sup\{k-l_2,0\}\leq a\leq\frac{d-1}{2}}\left(-\frac{2a+1}{d}\{(\nu_1+\rho_1)-(l_2+2a+2)\}\right.\\ &+ \begin{bmatrix}\mathbf{0}, & \mathbf{diag}_{\frac{1}{2}sup\{k-l_2,0\}\leq a\leq\frac{d-3}{2}}\left(\frac{d-2a-1}{d}\{(\nu_2+\rho_2)-(l_2+2a+2)\}\right. \\ 0, & \mathbf{0}\end{bmatrix}\end{aligned}$$

Going-down relations: $\mathcal{C}_{-;(+2)} : \tau_{(l_1,l_2)} \to \tau_{(l_1,l_2-2)}$ (2 case)

(v) If $\tilde{l}_2 \leq k$, we have $m' = m$ and

$$\mathcal{C}_{-;(+2)}\mathbf{s}^{(d)}_{[k-l_2,\cdots,d]}\Delta^{l_2} = \mathbf{s}^{(d+2)}_{[k-l_2+2,\cdots,d+2]}\Delta^{l_2-2} \cdot \Gamma_{-;(+2)}(\pi_{P_J,k};(l_1,l_2))$$

with some $\frac{1}{2}(\tilde{l}_1-k+3)\times\frac{1}{2}(\tilde{l}_1-k+3)$ intertwining matrix $\Gamma_{-;(+2)}(\pi_{P_J,k};(l_1,l_2))$, which is given by

$$\begin{aligned}\Gamma_{-;(+2)}(l_1,l_2) &= \begin{bmatrix}0, & \mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2+1)\leq a\leq\frac{d-1}{2}}\left(\frac{(d-2a)(d+1-2a)}{d+1)(d+2)}(k-\tilde{l}_2-2a)\right) \\ 0, & \mathbf{0}\end{bmatrix}\\ &+\mathbf{diag}_{\frac{1}{2}(k-\tilde{l}_2-1)\leq a\leq\frac{d-1}{2}}\left(\frac{(2a+2)(2a+3)}{(d+1)(d+2)}(\nu_J+\rho_J-\tilde{l}_2+d-2a)\right)\end{aligned}$$

(vi) If $\tilde{l}_2 \geq k+2$, we have $m' = m+1$ and

$$\mathcal{C}_{-;(+2)}\mathbf{s}^{(d)}_{[p_k(l_1),\cdots,d-p_k(l_2)]}\Delta^{l_2} = \mathbf{s}^{(d+2)}_{[p_k(l_1),\cdots,d+2-p_k(l_2)]}\Delta^{l_2-2} \cdot \Gamma_{-;(+2)}(l_1,l_2)$$

with some $(\frac{d+1}{2}) \times (\frac{d+3}{2})$ intertwining matrix $\Gamma_{-;(+2)}(l_1, l_2)$, which is given by

$$\Gamma_{-;(+2)}(l_1, l_2) = \begin{bmatrix} \mathbf{diag}_{0\le a \le \frac{d-1}{2}}(\frac{(d-2a)(d+1-2a)}{d+1)(d+2)}(k - \tilde{l}_2 - 2a)) \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{diag}_{0\le a \le \frac{d-1}{2}}(\frac{(2a+2)(2a+3)}{(d+1)(d+2)}(\nu_J + \rho_J - \tilde{l}_2 + d - 2a)) \end{bmatrix}$$

3 Embedding of the discrete series into P_J principal series as $(\mathfrak{g}, K)$ modules

Here we discuss an example. Th embedding of the holomorphic discrete series is well-known. So wi omit that here.

3.1 Successive composition of the contiguous relations

We have the contiguous eqaution:

$$\mathcal{C}^{(2a)}_{-,(+2)}\{\mathbf{s}^{(2a)}_{2a}\Delta^{k-2a}\} = \{\mathbf{s}^{(2a+2)}_{2a+2}\Delta^{k-2a-2}\} \cdot (\nu_J + \rho_J - k + 2a).$$

An equivalent equation for generic ν_J is

$$\mathbf{s}^{(2b)}_{2b}\Delta^{(k-2b)} = \frac{1}{\nu_J + \rho_J - k + 2b - 2}\mathcal{C}^{(2b-2)}_{-;(+2)}\mathbf{s}^{(2b-2)}_{2b-2}\Delta^{k-2b+2}.$$

Remark The successive application of the above equations gives

$$\mathbf{s}^{(2b)}_{2b}\Delta^{k-2b} = \prod_{i=0}^{b-1} \frac{1}{\nu_J + \rho_J - k + 2i} \cdot \mathcal{C}^{(2b-2)}_{-;(+2)} \cdots \mathcal{C}^{(0)}_{-;(+2)} \cdot \Delta^k..$$

3.2 Embedding of the large discrete series

Proposition E.2 Let $D^{(+,-)}_{(k,l)}$ be a large discrete series with Blattner parameter (k, l) $(k + l \ge 2, l < 0)$. Then there is a embedding of $(\mathfrak{g}, K)$ modules into the P_J principal series $\pi_{P_J,(\nu_J,e\otimes D^{\pm}_k}$ (if and) only if $\nu_J = -l$.

Proof) $\nu_J + \rho_J + k - 2a_{fin} - 2 = 0$, i.e., $\nu_J = 2a_{fin} - k = -l$. And $l = k - 2a_{fin}$ or $a_{fin} = (k-l)/2$. The assumption $l < 0$ and $k + l \geq 2$ implies that

$$k/2 < a_{fin} < k - 1.$$

Reamrk The vanishing of intertwining constant:

$\nu_J + \rho_J - k + 2i = 0$ is equivalent to $2a_{fin} - 2k + 2i-$, i.e., $i = k - a_{fin}$. This means that $1 \leq i = k - \frac{k-l}{2} = \frac{k+l}{2} < a_{fin}$.

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